Module 2: Properties of Integers

Problems on Mathematical Induction

1. Define the well ordering principle by Mathematical induction, prove that
$$1 + 2 + 3 + \cdots \ldots \ldots \ldots + n = \frac{1}{2}n(n+1), n \epsilon z^{+}.$$
2. Prove that $1^2 + 3^2 + 5^2 + \cdots \ldots + (2n-1)^2 = \frac{n(2n+1)(2n-1)}{3}$ by mathematical induction.
3. Prove that $1^2 + 2^2 + 3^2 + \cdots \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$ by mathematical induction.
4. By Mathematical Induction prove that
$$1 \times 3 + 2 \times 4 + \cdots \ldots \ldots + n(n+2) = \frac{n(n+1)(2n+7)}{b}.$$
5. By Mathematical Induction prove that for positive integer 'n' the number $11^{n+2} + 12^{2n+1}$ is divisible by 133.
6. Prove by mathematical induction for every positive integer n, 5 divides $(n^5 - n)$.
7. Prove that $4n < (n^2 - 7)$ for all positive integers $n \geq 6$.
8. Define the well ordering principle. By mathematical induction prove that $(n!) \geq 2n - 1$ for all positive integers.
9. Prove for any positive integer n, $\sum_{i=1}^{n} \frac{F_{i-1}}{2^i} = 1 - \frac{F_{(n+2)}}{2^n}$, F_n denotes Fibonacci number.
10. For the Fibonacci sequence $F_0, F_1, F_2, \ldots \ldots$ prove that $F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$.
11. If $F_0, F_1, F_2, \ldots \ldots$ Are Fibonacci numbers, prove that $\sum_{i=0}^{n} F_i^2 = F_n \times F_{n+1}$.
12. For the Lucas sequence $L_0, L_1, L_2, \ldots \ldots$ prove that $L_n = \left[\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$.

Problems on recurrence relations

13. Obtain the recurrence definition for the sequence in each of the following cases:
$$a_n = 5n, a_n = 3n + 7, a_n = n^2, a_n = 2 - (-1)^n$$
14. Let $a_0 = 1, a_1 = 2, a_2 = 3$ and $a_n = a_{n-1} + a_{n-2} + a_{n-3}$ for $n \geq 3$, prove that $a_n \leq 3^n$ for all positive integers.

Problems on permutations and combinations

15. Find the number of permutations of the letters of the word 'MASSASAUGA'. In how many of these all four A's are together? How many pf them begin with S's?
16. Find the number of ways of arrangement of the letters of the word 'TALLAHASSEE' which have no adjacent A's.
17. Find the number of permutations of the letters of the word 'ENGINEERING' such that: (i) All the E's are together (ii) Arrangements begin with N (iii) All the vowels are adjacent.

18. How many arrangements are there for all the letters in the word "SOCIOLOGICAL". In how many of these arrangements (i) A and G are adjacent (ii) All vowels are adjacent.
19. How many positive integers n can we form using the digits 3, 4, 4, 5, 5, 6, 7 if we want n to exceed $5{,}000{,}000$?
20. How many words can be made with or without meaning from the letters of the word 'STATISTICS'? In how many of these 'a' and 'c' are adjacent? In how many vowels are together?
21. In how many ways can we distribute 7 apples and 5 oranges among 3 children such that each child gets al least one apple and one orange?
22. In how many ways one can distribute 8 identical marbles in 4 distinct containers so that i) no container is empty ii) the fourth container has an odd number of marbles in it.
23. A woman has 11 close relations and wishes to invite 5 of them to dinner. In how many ways can she invite them if (i) there is no restriction on her choice, (ii) 2 persons will not attend separately, (iii) 2 persons will not attend together.

Problems on Binomial and Multinomial Theorem

24. Find the coefficient of $a^2b^3c^2d^5$ in the expansion of $(a+2b-3c+2d+5)^{16}$.
25. Find the coefficient of x^3y^8 in the expansion of $(2x-y)^{11}$.
26. Find the coefficient of x^9y^3 in the expansion of $(2x-3y)^{12}$.
27. Find the coefficient of xyz^2 in the expansion of $(2x-y-z)^4$.
28. Find the coefficient of a^5b^2 in the expansion of $(2a-3b)^7$.
29. Find the coefficient of x^5y^2 in the expansion of $(x+y)^7$.
30. Find the coefficient of x^{12} in the expansion of $x^3(1-2x)^{10}$.